In this work, we use the nonlinear Time Dependent Ginzburg - Landau equations (TDGL) to theoretically study the magnetization of a hollow square cylinder (square ring geometry) in the presence of an external magnetic field applied perpendicularly to the ring plane. We calculate the spatial distribution of the superconducting electron density and the phase of the superconducting order parameter using the numerical method based on the technique of gauge invariant variables. We obtain isothermal magnetization curves for different values of the Ginzburg–Landau parameter ($\kappa$) and temperature.

Keywords: Superconductor; Ginzburg–Landau; mesoscopics.

En este trabajo usamos las ecuaciones no lineares Ginzburg-Landau dependientes del tiempo (TDGL) para estudiar teoricamente la magnetización de un cilindro cuadrado hueco (anillo cuadrado) en la presencia de un campo magnético aplicado perpendicular al plano del anillo. Calculamos la distribución espacial de la densidad de electrones superconductores y la fase del parámetro de orden usando el método numérico basado en la técnica de variables de calibre. Obtenemos curvas isotérmicas de magnetización para diferentes valores del parámetro Ginzburg-Landau ($\kappa$) y temperatura.

Descriptores: Superconductor; Ginzburg–Landau; mesoscópico.

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1. Introduction

In the last decade the study of mesoscopic superconducting rings and disks has attracted a lot of attention; in particular, there is an interest in the quantum coherence effects in superconducting rings and their arrays for potential applications in quantum computing. Technological advancements in nanostucturing has made it possible to fabricate and study superconducting samples with sizes comparable to the coherence length ($\xi$) and/or the London penetration depth ($\lambda$). The confinement effects have been clearly demonstrated experimentally in nanosized superconducting square loops [1], disks and ring [2]. The fluxoid quantization in these systems has been known for many years ago and the properties of mesoscopic superconductors have drawn growing interest in both experimental and theoretical works [3] [4]. Several results about the quantization of flux in mesoscopic samples have been presented. Baelus et al. [5] studied the superconducting state of a thin superconducting disk with a hole. Morelle et al. [6] studied the nucleation of superconductivity in a mesoscopic loop of varying width; they found a parabolic background of $T_c$ with periodic oscillations found for the thinnest loops. Sardella et al. [7] investigated the dynamics of vortices in a square mesoscopic superconductor for several values of temperature; they demonstrated that the temperature has a crucial role in the formation of the vortex state. Kato et al. [8] studied the effects of the surface boundary on the magnetization process in type II superconductors.

In the present work we examine the magnetization curves and the nucleation process as a function of the external magnetic field, applied perpendicularly to the ring plane for several values of temperature for a mesoscopic square ring. By numerically resolving the TDGL equations, and by using the numerical method based on the technique of gauge invariant variables ($U$-$\psi$ method), we observe that the vortex entrance into the sample corresponds to a phase transition with a four-fold symmetry. The magnetization curves are qualitatively constant for several values of the Ginzburg - Landau parameter ($\kappa = \lambda/\xi$).

2. Time dependent Ginzburg–Landau equations

We theoretically study the magnetization of a hollow square cylinder. From now on, we will refer to this geometry as a square ring. The TDGL equations which govern the superconductivity order parameter $\psi$, and the vector potential $\vec{A}$ are given by

\[ \frac{\partial \psi}{\partial t} = -\eta \left[ (-i\nabla - \vec{A})^2 \psi + (1-T) \left( |\psi|^2 - 1 \right) \psi \right] + \tilde{f} \quad (1) \]

\[ \frac{\partial \vec{A}}{\partial t} = (1-T) \Re \left[ \psi (-i\nabla - \vec{A}) \psi \right] + \kappa^2 \nabla \times \nabla \times \vec{A} \quad (2) \]

lengths have been scaled in units of $\xi(0)$, time in units of $t_0 = \pi \hbar / (96 K_B T_c)$, the external applied magnetic field $H_e$ in units of $H_{c2}(0)$, the potential vector $\vec{A}$ in units of $H_{c2}(0)\xi(0)$, temperatures in units of $T_c$, $\eta$ is a positive constant, $\tilde{f}$ a random force simulating thermal fluctuations. $\Re$ indicates the real part. The boundary condition for superconductor/vacuum interphase is given by

\[ \hat{n} \cdot (-i\nabla - \vec{A}) \psi = 0 \quad (3) \]

where $\hat{n}$ is the unitary vector perpendicular to the surface of the superconductor. This boundary condition implies that the
perpendicular component of the superconducting current is equal to zero at the surface. The magnetization \( M_z \) is:

\[
M_z = \frac{\int [B_z(x, y) - H_e]dxdy}{4\pi \int dxdy}
\]  

(4)

3. Numerical method

The full discretization of the TDGL equations can be found in more detail in Ref. [9]. The widely used \( U - \psi \) method is described in detail by Gropp et al. [10]. Complex link variables \( U^x \) and \( U^y \) are introduced to preserve the gauge-invariant properties of the discretized equations. \( U^x \) and \( U^y \) are related to \( A \) by

\[
U^x (x, y, t) = \exp \left( -i \int_{x_0}^x A_x (\xi, y, t) d\xi \right)
\]

(5)

\[
U^y (x, y, t) = \exp \left( -i \int_{y_0}^y A_y (x, \eta, t) d\eta \right)
\]

(6)

\((x_0, y_0)\) is an arbitrary point. The link variable method is used since a better numerical convergence is obtained in high magnetic fields [11]. The TDGL equations (1)-(2) can be written in the following form:

\[
\frac{\partial \psi}{\partial t} = \bar{U}_x \frac{\partial^2 (U_x \psi)}{\partial x^2} + \bar{U}_y \frac{\partial^2 (U_y \psi)}{\partial y^2}
\]

\[
+ (1 - T) \psi \left( 1 - |\psi|^2 \right) + \bar{f}
\]

(7)

\[
J_{s0} = (1 - T) \left[ \bar{U}_\alpha \psi \frac{\partial (U_\alpha \psi)}{\partial \alpha} \right]
\]

(8)

where \( \alpha = (x, y) \), and \( Im \) indicates the imaginary part. We used this method to obtain our results. The outline of this simulation procedure is as follows: the sample is divided into a rectangular mesh consisting of \( N_x \times N_y \) cells, with mesh spacing \( a_x \times a_y \). In our simulation we use the simple Euler method with 100 steps. \( \Delta t = 0.015 \), spacing \( a_x = a_y = 0.5 \), grid size 50 \( \times \) 50 with 46 \( \times \) 46 for the hole. \( H_e \) is linearly increased with time from 0 to 1, with small intervals of \( \Delta H = 10^{-6} \); \( \bar{f} \) is treated as a vertex variable selected at each mesh point from a Gaussian distribution with standard deviation \( \sigma \) given by:

\[
\sigma = \sqrt{(\pi E_0/6\Delta t)(T/T_0)}
\]

(9)

where \( E_0 \) is the ratio of the thermal energy to the free energy of a vortex, see Ref. [8]. For example, in a superconductor with \( T_c = 10 \) K, and \( \xi(0) = 100 \) A, the sample size is \( \sim 1 \mu m \times 1 \mu m \), the characteristic time \( t_0 = 10^{-13} \) s, and the parameter \( E_0 = 10^{-7} \).

4. Results

Fig. 1(a) shows the sample geometry. The phase of the order parameter is plotted for a superconductor square ring; values of the phase close to zero are given by dark gray regions and those close to \( 2\pi \) are given by light gray regions. The phase allows one to determine the number of vortices in a given region by counting the phase variation in a closed path around this region. If the vorticity in the region is \( L \), then the phase changes by \( \Delta \phi = 2\pi L \). The number of vortices increases into the ring for the same applied magnetic field when the temperature is increased. Fig. 1 (b-d) shows that the number of vortices increase with the temperature for the same applied magnetic field; this dependence is better illustrated in Fig. 4 (a-c).

![Figure 1](image1.png)

**Figure 1.** (a) the sample geometry is depicted. Phase of the order parameter for (b) \( T = 0 \), (c) \( T = 0.375 \), (d) \( T = 0.875 \) respectively, at \( H_e = 0.15 \), for \( \kappa = 2 \). Dark and bright regions represent values of the modulus of the order parameter (as well as \( \Delta \phi/2\pi \), from 0 to 1).

![Figure 2](image2.png)

**Figure 2.** Magnetization curves as a function of the applied magnetic field for \( T = 0.5 \) and (a) \( \kappa = 0.5 \), (b) \( \kappa = 0.7 \), (c) \( \kappa = 1.0 \), (d) \( \kappa = 2.0 \), respectively.
The resulting magnetization curve exhibits a series of maxima (Fig. 2). These maxima correspond to points of maximal penetration of the magnetic field for (a) \( \kappa = 0.5 \), (b) \( \kappa = 0.7 \), (c) \( \kappa = 1.0 \), and (d) \( \kappa = 2.0 \), respectively. The width of the ring we considered here is small enough to ensure that it is superconducting even when more than one flux enters the ring for several temperatures used in the simulation. The different \( \kappa = 0.5, 0.7, 1.0 \) and 2.0, studied represent both type I and type II superconductors, respectively. It is observable that the qualitative behaviors of the magnetization curves are almost insensitive to \( \kappa \) changes; this is due to the width of the ring, which leads to constant screening supercurrents inside the sample, thus rising to a constant magnetization in this interval of magnetic field.

By using equation (4) we have evaluated the magnetization for several values of temperatures ranging from \( T = 0 \) to \( T = 0.85 \). The result for the magnetization as a function of the applied external field is depicted in Fig. 3. As can be seen, the magnetization exhibits a series of peaks. Each one of these peaks signals a vortex entrance. We can also notice that the magnetic field, in which the magnetization is non-vanishing, varies with the temperature. At zero temperature, the system has the longest magnetization values. The magnetization does not scale with the temperature; that is, the quantitative behavior of this quantity is distinct for different temperatures although it shows the same qualitative behavior.

In Fig. 4 (a-c), we illustrate the number of vortices \( N \) for several temperatures as a function of \( H_\text{e} \), clearly illustrating that the increase of temperature diminishes the magnetic field in which the entrance of the first vortices occurs. We observe that the entrance of the first vortices corresponds to a phase transition from \( L = 0 \) to \( L = 4 \) vortex state. In general, we have transitions from \( L \) to \( L + 4 \) vortex state, corresponding to a phase transition with fourfold symmetry; this does not vary for different temperatures. We compare this result with the phase transition corresponding to thin superconducting disks with a hole. The system makes a transition \( L \) to \( L + 1 \), indicating the entrance of an extra vortex. The effect of the position and the size of the hole on the vortex configuration is very important [5]. The sample geometry is very important in the nucleation process.

In conclusion, we have solved the TDGL equation of both type I (\( \kappa < 1/\sqrt{2} \)) and type II (\( \kappa > 1/\sqrt{2} \)) ring superconductors in the presence of an external magnetic field applied perpendicularly to the ring plane. We notice that the magnetic field in which the magnetization is non-vanishing varies with the temperature and with the Ginzburg - Landau parameter \( \kappa \). The entrance of the vortices into sample corresponds to a phase transition with the four-fold symmetry. The nucleation process occurs from \( L \) to \( L + 4 \) when the external magnetic
field is increased from $H_e = 0$ to $H_e = 0.30$; this nucleation process is independent of the temperature for this interval of magnetic fields. From the magnetization curves we can obtain the first field for flux penetration $H_{C1}$. We find an increase of $H_{C1}$ with a decreasing $\kappa$.

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